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SOME NEW RESULTS IN REGENERATIVE PROCESS THEORY

by

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1. Introduction

The concept of a regenerative stochastic process has become an important tool in applied probability. From its theoretical origins in work of DOEBLIN (1938) and SMITH (1955), it has grown to play a central role in the analysis of, for example, queueing systems in light traffic and simulation output analysis; see IGLEHART (1971) and CRANE and LEMOINE (1978).

In the Markov chain setting, regenerative analysis has simplified many complicated analytical arguments associated with the limit theory of such processes. A significant step in this direction occurred with the recent papers by ATHREYA and NEY (1978) and NUMMELIN (1978), who showed that regenerative process theory is applicable to a significantly larger class of Markov chains than previously known.

In this report, we shall investigate various properties of discrete-time regenerative processes $\{X_n\}$. We start, in Section 2, by defining the concept of a regenerative stochastic process. In contrast to many other treatments (e.g., ÇINLAR (1975)), we make no requirement that regeneration times be stopping times with respect to the process fields, or, in fact, that they even be measurable with respect to the process. We also generalize the concept of a regenerative process to allow m-dependence between regenerative blocks; we call this class of processes weakly regenerative. This

class turns out to be useful in treating general state space Markov chains; see [19] and [20].

In Section 3, ergodic theory for weakly regenerative processes is investigated. We prove strong laws for partial sum processes, and show that there exists an "ergodic" measure π which captures all the "steady-state" information of the process $\{X_n\}$. We also obtain a.s. weak convergence for the empirical measure of a weakly regenerative process, and show that the "renewal paradox" continues to hold in this generalized setting.

Section 4 is concerned with construction of a stationary process from the ergodic measure of Section 3. Total variation convergence of the measures

$$\sum_{k=0}^{n-1} P\{X_k \in \cdot\}/n$$

to π is studied, in the regenerative setting, in Section 5. Furthermore, rates of convergence, in terms of the moments of the inter-regeneration times, are obtained. These ideas are applied in Section 6 where it is shown that all regenerative processes are strong mixing, and that regenerative processes are uniformly strong mixing (ϕ-mixing) under a simple sufficient condition. These results allow elementary proofs of some classical mixing results of DAVYDOV (1973) for Markov chains, and improve the results in the sense that estimates for the mixing constants can be obtained. Section 7 studies the

central limit theorem (CLT) for weakly regenerative processes; this result can be used to obtain a new CLT for Markov chains; see [20]. We also relate this CLT to a CLT for the corresponding stationary process, constructed in Section 4.

Finally, in Section 8, we investigate the splitting property of regeneration times (see Jacobsen (1974) for the definition of a splitting time) for a special class of Markov chains. A regenerative process is said to be strongly regenerative if the regeneration times are measurable with respect to the σ-field generated by the process. For the class of Markov chains considered, we are able to show that all strong regeneration times are splitting times. This result allows us to completely describe the strongly regenerative Markov chains in the class, and shows that when such a Markov chain is strongly regenerative, then the regeneration times can be chosen to be stopping times with respect to the process fields.

2. Definitions and Preliminaries

Let $\{X_n: n \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \underline{G}, P)$, and taking values in a measurable space $(E, \underline{E})$. A $\underline{G}$-measurable random variable (r.v.) $T: \Omega \to Z^+ = \{0,1,\ldots\}$ is called a <u>random time</u>. For any two random times T_1, T_2 such that $T_1 \leq T_2$, we define $\underline{F}_{T_1}^{T_2}$ to be the σ-field generated by all events of the form $A \cap \{T_2 - T_1 = k\}$, where

$$A \in \underline{F}_{T_1}^{T_1+k} \equiv \underline{B}(X_{T_1}, X_{T_1+1}, \ldots, X_{T_1+k}).$$

(2.1) DEFINITION. A process $\{X_n: n \geq 0\}$ is said to be <u>regenerative</u> if there exist random times $0 = T_0 < T_1 < T_2 < \cdots$ such that:

i) the σ-fields $\underline{H}_n = \underline{F}_{T_n}^{T_{n+1}-1}$ are independent, for $n \geq 0$,

ii) the σ-fields $\underline{H}_n$ are identically distributed for $n \geq 1$, in the sense that

$$P\{(X_{T_n}, \ldots, X_{T_n+k-1}) \in B;\ \tau_n = k\}$$
$$= P\{(X_{T_1}, \ldots, X_{T_1+k-1}) \in B;\ \tau_1 = k\}$$

for each B in the product σ-field $\underline{E}^k = \underline{E} \times \cdots \times \underline{E}$ (k times), where $\tau_n = T_{n+1}-T_n$.

The random times $T_1, T_2, \ldots$ are called <u>regeneration times</u> for the process $\{X_n: n \geq 0\}$. One desirable property of this definition, in analogy with the case of independent and identically distributed (i.i.d.) r.v.'s, is that the process $\{Z_n: -\infty < n \leq 0\}$ inherits the regenerative property (with the obvious generalization of (2.1) to time parameter set $\{k: -\infty < k \leq 0\}$), where $Z_n = X_{-n}$.

(2.2) PROPOSITION. The regenerative property is preserved under time reversal.

Proof. For the process $\{Z_n\}$, we consider the random times S_j given by $-T_j+1$, and put $S_0 = 1$. It is easily checked that the generating sets of $\underline{F}_{S_j}^{S_{j-1}-1}$ are precisely those of $\underline{F}_{T_{j-1}}^{T_j-1}$. This immediately yields the conclusion that $\underline{F}_{S_{j+1}}^{S_j-1}$ are independent for $j \geq 0$, and identically distributed for $j \geq 1$. ∎

We remark that one of the classical definitions of a regenerative process requires that the random times T_j be stopping times with respect to the family of increasing σ-fields $\{\underline{F}_0^n: n \geq 0\}$; see, for example, ÇINLAR (1975), p. 298. Such a definition, in contrast to ours, does not have the time reversal property. Furthermore, it is clearly a limitation to require that the T_j be $\underline{F}_0^\infty$ measurable.

(2.3) EXAMPLE. Let $\{Z_n: n \geq 0\}$ be a finite state, irreducible Markov chain on $\{0, 1, ..., m\}$. Then, $X_n = I_{\{0\}}(Z_n)$ $(I_A(x) = 1(0)$ if $x \notin A(x \in A))$ is a regenerative process under (2.1), and yet, in general, there exist no $\underline{F}_0^\infty$ measurable regeneration times for X_n.

As we shall see in Section 5, $\underline{F}_0^\infty$ measurability of the T_j's has some important consequences, motivating the following definition.

(2.4) **DEFINITION.** The process $\{X_n: n \geq 0\}$ is said to be <u>strongly regenerative</u> if $\{X_n\}$ is regenerative and the regeneration times T_k are $\underline{F}_0^{\infty}$ measurable.

Applications to general state space Markov chains in [19] require that we weaken the regenerative property somewhat.

(2.5) **DEFINITION.** The process $\{X_n\}$ is said to be <u>weakly regenerative of order m</u> if there exist random times T_j such that:

i) the σ-fields $\underline{H}_n = \underline{F}_{T_n}^{T_{n+1}-1}$ are m-dependent for some m, i.e., the σ-fields $\underline{H}_n$ and $\underline{H}_{n+j}$ are independent for $j \geq m+1$,

ii) for $n \leq 1$ and $j_0 < \cdots < j_\ell$

$$P\{(X_{T_n}, \ldots, X_{T_n+k}) \in B;\ \tau_n = j_0, \ldots, \tau_{n+\ell} = j_\ell\}$$

$$= P\{(X_{T_1}, \ldots, X_{T_1+k}) \in B;\ \tau_1 = j_0, \ldots, \tau_{n+\ell+1} = j_\ell\}$$

for each B in the product σ-field $\underline{E}^k$, where $\tau_n = T_{n+1}-T_n$.

(2.6) **PROPOSITION.** A regenerative process $\{X_n\}$ is weakly regenerative of order 0.

Proof. We need only check (2.5) (ii). Let $A \in \underline{E}^j$ and $B \in \underline{E}^k$, and consider

$$P\{(X_{T_n}, \ldots, X_{T_n+k+j-1}) \in A \times B;\ \tau_n = j,\ \tau_{n+1} = k\}$$

$$= P\{(X_{T_n}, \ldots, X_{T_n+j-1}) \in A;\ \tau_n = j\}$$

$$\cdot\ P\{(X_{T_{n+1}}, \ldots, X_{T_{n+1}+k-1}) \in B;\ \tau_{n+1} = k\}$$

$$= P\{(X_{T_1}, \ldots, X_{T_1+j-1}) \in A;\ \tau_1 = j\}$$

$$\cdot\ P\{(X_{T_2}, \ldots, X_{T_2+k-1}) \in B;\ \tau_2 = k\}$$

$$= P\{(X_{T_1}, \ldots, X_{T_1+k+j-1}) \in A \times B;\ \tau_1 = j,\ \tau_2 = k\}\ ,$$

the first and third equalities by (2.1)(i), the second by (2.1)(ii). An easy induction argument extends the above equation to the class of rectangles generating (2.5)(ii), proving the result. ∎

We now examine some elementary properties of weakly regenerative processes.

(2.7) **PROPOSITION.** Let f_k be a sequence of real-valued functions such that f_k is $\underline{E}^k$ measurable. Then, $Y_n = f_{\tau_n}(X_{T_n}, \ldots, X_{T_n+\tau_n-1})$ is $\underline{H}_n$ measurable. Hence, the r.v.'s $\{(Y_n, \tau_n): n \geq 1\}$ are identically distributed, and m-dependent.

Proof. We write Y_n as

$$(2.8) \qquad Y_n = \sum_{k=0}^{\infty} f_{k+1}(X_{T_n}, \ldots, X_{T_n+k})\, I_{\{\tau_n = k+1\}}$$

(note that (2.8) is a finite sum for each $\omega \in \Omega$), and observe that each summand in (2.8) is $\underline{H}_n$ measurable, so that Y_n is $\underline{H}_n$ measurable. ∎

Certain renewal arguments will demand the following result.

(2.9) **PROPOSITION.** Suppose that $\{X_n\}$ is regenerative and that $g \in b\underline{F}^{\infty}$, where $b\underline{F}^{\infty}$ is the class of real-valued bounded $\underline{F}^{\infty}$ measurable random variables. Then,

i) $E\{g(V_{T_1+n});\ T_2 = T_1+k\} = Eg(V_{T_1+n-k})\, P\{\tau_1 = k\}$

where $1 \leq k \leq n$, and $V_j = (X_j, X_{j+1}, \ldots)$.

ii) $E\{g(V_{k+n});\ T_{\ell(k)+1} = k+j \,|\, \underline{F}_0^k\}$

where $\ell(n) = \max\{k: T_k \leq n\}$, and $0 \leq j \leq n$.

Proof. For (i), write

$$E\{g(V_{T_1+n});\ T_2 = T_1+k\}$$

$$= E\{g(V_{T_2+n-k});\ \tau_1 = k\}$$

and observe that $g(V_{T_2+n-k})$ is measurable with respect to $\vee_{j=2}^{\infty} \underline{H}_j$, the minimal σ-field generated by $\underline{H}_j$, $j \geq 2$. The independence of $\underline{H}_1$ from this σ-field, together with (2.5)(ii), yields (i). For (ii), take $W \in b\underline{F}_0^k$ and decompose on $\ell(k)$ as follows:

$$E\{Wg(V_{k+n}); T_{\ell(k)+1} = k+j\}$$

$$= \sum_{j=0}^{k} \sum_{\ell=j}^{k} E\{Wg(V_{k+n}); T_j = \ell, \tau_j = k+j-\ell\}$$

$$= \sum_{j=0}^{k} \sum_{\ell=j}^{k} E\{Wg(V_{T_{j+1}+(n-j)}); T_j = \ell; \tau_j = k+j-\ell\} .$$

Here, $WI_{\{T_j=\ell;\tau_j=k+j-\ell\}}$ is $\vee_{i=0}^{j} \underline{H}_i$ measurable, whereas $g(V_{T_{j+1}+(n-j)})$ is $\vee_{i=j+1}^{\infty}$ measurable, and the resulting independence proves (i) (use the defining relation for conditional expectation). ∎

3. Ergodic Theory for Weakly Regenerative Processes

A weakly regenerative process is said to be positive recurrent if $E\tau_1 < \infty$ and null recurrent otherwise. In the remainder of this section, we assume that $\{X_n\}$ is a positive recurrent weakly regeneration process.

Before proceeding, we need some notation. Let $b\underline{E}$ be the class of bounded real-valued $\underline{E}$ measurable functions, and let $\underline{E}^+$ be the cone of non-negative $\underline{E}$ measurable functions. For $\underline{E}$ measurable real-valued f, put

$$Y_n(f) = \sum_{k=T_n}^{T_{n+1}-1} f(X_k) \quad .$$

The following theorem extends Theorem 7 of SMITH (1955) for cumulative processes.

(3.1) THEOREM. **Suppose either that $f \in \underline{E}^+$ or $EY_1(|f|) < \infty$. Then,**

$$\sum_{k=0}^{n} f(X_k)/n \to EY_1(f)/E\tau_1 \qquad \text{a.s.} \tag{3.2}$$

Proof. First, observe that by Proposition 2.7, $\{(Y_n(f);\ \tau_n);\ n \geq 1\}$ is an identically distributed m-dependent sequence. For $f \in \underline{E}^+$, the strong law of large numbers implies that

$$\sum_{k=0}^{\ell} Y_{(m+1)k+j}(f)/\ell \to EY_1(f) \qquad \text{a.s.} \tag{3.3}$$

as $\ell \to \infty$, for $j = 1, \ldots, m+1$. Averaging (3.3) over j, and applying the result to both $Y_i(f)$ and $Y_i(1)$, shows that

$$\sum_{k=0}^{\ell} Y_k(f)/T_\ell \to EY_1(f)/E\tau_1 \qquad \text{a.s.} \tag{3.4}$$

The non-negativity of $f \in \underline{E}^+$ implies that

$$(3.5) \qquad \sum_{k=0}^{\ell(n)} Y_k(f)/T_{\ell(n)+1} \le \sum_{k=0}^{n} f(X_k)/n \le \sum_{k=0}^{\ell(n)+1} Y_k(f)/T_{\ell(n)} .$$

Using (3.4) on the end terms of (3.5) yields (3.2) for $f \in \underline{E}^+$. For f satisfying $EY_1(|f|) < \infty$, split f into $f^+(x) = \max\{f(x),0\}$ and $f^-(x) = \max\{-f(x),0\}$. Since (3.2) holds for f^+ and f^-, we obtain

$$(3.6) \qquad \sum_{k=0}^{n} f(X_k)/n \to (EY_1(f^+) - EY_1(f^-))/E\tau_1 .$$

If $EY_1(|f|) < \infty$, we can combine the expectations on the right side of (3.6) as $EY_1(f)/E\tau_1$. ∎

The following example shows that convergence may not occur under $E|Y_1(f)| < \infty$.

(3.7) EXAMPLE. Let $\{Z_k: k \ge 0\}$ be a sequence of non-negative independent and identically distributed (i.i.d.) r.v.'s with $EZ_0 = \infty$. Put $X_{2k} = Z_k$, and $X_{2k+1} = -Z_k$. Then, $T_k = 2k$ and $Y_k(g) = 0$ for $g(x) = x$. But

$$\sum_{j=0}^{2k} Y_j(g)/2k = Z_k/2k \to \infty \neq EY_1(g)/E\tau_1 .$$

The convergence of Z_k/k to infinity is implied by $\sum_{k=0}^{\infty} P\{Z_k > k\} = \infty$, and the Borel-Cantelli converse.

Let π be the set function on $\underline{E}$ defined by the formula

$$\pi(A) = E\{\sum_{k=T_1}^{T_2-1} I_A(X_k)\}/E\tau_1 . \tag{3.8}$$

It is easy to show that π is countably additive and thus a probability measure. We shall use the notation πf to denote $\int f(y)\, \pi(dy)$. The following "representation" theorem for the "ergodic" measure π is valid.

(3.9) PROPOSITION. If πf is well-defined, then

$$\pi f = EY_1(f)/E\tau_1 . \tag{3.10}$$

Proof. By definition of π, (3.10) clearly holds for all simple functions f. For $f \in b\underline{E}$, f can be uniformly approximated by simple functions f_n such that

$$\|f_n - f\| = \sup\{|f_n(x) - f(x)|: \ x \in E\} < 1/n ,$$

so that

$$\left|\pi f - EY_1(f)/E\tau_1\right|$$

$$= \left|\pi(f-f_n) + (\pi f_n - EY_1(f_n)/E\tau_1) + EY_1(f-f_n)/E\tau_1\right|$$

$$\to 2\|f-f_n\| \leq 2/n \quad ,$$

and hence (3.10) holds for $f \in b\underline{E}$. For $f \in \underline{E}^+$, we approximate f by bounded functions f_n increasing to f, and use monotone convergence. For a general f, write $f = f^+ - f^-$, and observe that (3.10) is valid for f^+ and f^-. ∎

Consider the sequence of empirical probability measures defined by

$$M_n(A,\omega) = \sum_{k=0}^{n} I_A(X_k(\omega))/(n+1) \ .$$

Our next theorem concerns the a.s. weak convergence of $M_n(\cdot,\omega)$.

(3.11) THEOREM. Suppose that $\underline{E}$ is the Borel σ-field of a separable metric space E. Then,

$$M_n(\cdot,\omega) \Longrightarrow \pi(\cdot) \qquad \text{a.s.}$$

where $\Longrightarrow$ denotes weak convergence of probability measures.

Before proceeding with the proof, we need to discuss the notion of weak convergence. It is known (see BILLINGSLEY (1968), p. 12) that $P_n \Longrightarrow P$ is equivalent to requiring tht $P_n(A) \to P(A)$ for each A such that $P(\partial A) = 0$, where ∂A is the boundary of A. For separable metric spaces, a smaller class of sets A characterizes weak convergence. Let $\{s_n\}$ be a set of points dense in E, and put $B(s_n,\varepsilon) = \{y: \rho(s_n,y) < \varepsilon\}$, where ρ is the metric on E. Then, observe that $\partial B(s_n,\varepsilon) \subset \{y: \rho(s_n,y) = \varepsilon\}$, and hence, for fixed n, the boundaries are disjoint in ε. Thus, for each n, one can find a sequence of numbers $\varepsilon_{mn} \downarrow 0$ such that $P(\partial B(s_n,\varepsilon_{mn})) = 0$ for all m. Let $\Lambda(P)$ be the class of all finite intersections of sets $B(s_n, \varepsilon_{mn})$.

(3.12) **LEMMA.** **If P_n, P are probability measures on a separable metric space E, then $P_n \Longrightarrow P$ is equivalent to $P_n(G) \to P(G)$ for all $G \in \Lambda(P)$.**

<u>Proof</u>. First, since $\partial(G_1 \cap G_2) \subset (\partial G_1) \cup (\partial G_2)$, it follows that $P(\partial G) = 0$ for all $G \in \Lambda(P)$ and thus $P_n \Longrightarrow P$ implies that $P_n(G) \to P(G)$ for all $G \in \Lambda(P)$. Conversely, observe that for each $x \in E$ and $\varepsilon > 0$, there exists m,n such that $B(s_n,\varepsilon_{mn}) \subset B(x,\varepsilon)$ and thus Corollary 1, page 14, of [3] applies, completing the proof. ∎

Note that $\Lambda(P) = \cup_{i=1}^{\infty} \Lambda_i(P)$, where $\Lambda_i(P)$ is the class of all intersections of i sets of the form $B(s_n,\varepsilon_{mn})$. Since each $\Lambda_i(P)$ is countable, $\Lambda(P)$ is countable.

Proof of Theorem 3.11. Since $\Lambda(P)$ is countable, it follows that $\{\omega: M_n(\cdot,\omega) \Rightarrow \pi(\cdot)\}$ is $\underline{G}$-measurable, by Lemma 3.12. Furthermore, this set has probability 1, since

$$P\{\omega: M_n(G,\omega) \to \pi(G) \quad \text{for all} \quad G \in \Lambda(\pi)\} = 1 \quad ,$$

using Theorem 3.1 and Proposition 3.9. ∎

(3.13) COROLLARY. Let E be a separable metric space, and let bC be the class of $\underline{E}$ continuous functions. Then,

$$P\{\omega: \sum_{k=0}^{n} f(X_k(\omega))/n \to \pi f, \quad \text{for all} \quad f \in bC\} = 1 \quad .$$

Thus, for weakly regenerative processes, one can obtain simultaneous convergence in (3.1) over a large class of functions, namely bC (this simultaneous convergence can not be extended to $b\underline{E}$, except when π is atomic).

In addition to strong laws such as (3.1), weakly regenerative processes also have behavior characteristic of the so-called "renewal paradox."

(3.14) PROPOSITION. Suppose either that $f \in \underline{E}^+$ or $E\tau_1 Y_1(|f|) < \infty$. Then,

$$(3.15) \qquad \sum_{k=0}^{n} Y_{\ell(k)}(f)/n \to E\tau_1 Y_1(f)/E\tau_1 \qquad \text{a.s.}$$

Proof. For $f \in \underline{E}^+$, observe that

$$\sum_{k=0}^{T_{\ell(n)}-1} Y_{\ell(k)}(f)/T_{\ell(n)} = \sum_{k=0}^{\ell(n)-1} \sum_{j=T_k}^{T_{k+1}-1} Y_{\ell(j)}(f)/T_{\ell(n)}$$

$$= \sum_{k=0}^{\ell(n)-1} \tau_k Y_k(f)/T_{\ell(n)} \to E\tau_1 Y_1(f)/E\tau_1 \quad .$$

a.s. (to deal with the m-dependence, average as in the proof of Theorem 3.1). Now, repeat the reasoning of (3.5) through (3.6) to complete the proof. ∎

In particular, the "time-averaged" length of the epoch $T_{\ell(n)+1} - T_{\ell(n)}$ is $E\tau_1^2/E\tau_1$, as in the regenerative case.

4. Stationary Regenerative Processes

Associated with every weakly regenerative process is a closely related stationary process; see BROWN and ROSS (1972) for a discussion in the regenerative case. We start by proving a strong law for functionals of the form $g(X_n, \ldots, X_{n+\ell})$, where g is a real-valued

$\underline{E}^{\ell+1}$ measurable function. In preparation for stating the result, we extend the notation $Y_n(f)$ from scalar functions f to functions g of a vector argument:

$$Y_n(g) = \sum_{k=T_n}^{T_{n+1}-1} g(X_k, \ldots, X_{k+\ell}).$$

(4.1) **PROPOSITION.** Suppose either that $g \in (\underline{E}^{\ell+1})^+$ or $EY_1(|g|) < \infty$. Then,

$$\sum_{k=0}^{n} g(X_k, \ldots, X_{k+\ell})/n \to EY_1(g)/E\tau_1 \qquad \text{a.s.} \tag{4.2}$$

Proof. Let $U_k = (X_k, \ldots, X_{k+\ell})$, and recall that X_k is weakly regenerative of order m, with respect to some sequence of random times T_j. It is easily verified that U_k is then weakly regenerative, of order $m+\ell$, with respect to the same times T_j. The limit theorem (4.2) is then obtained as an immediate consequence of Theorem 3.1. ∎

The following result shows that weakly regenerative processes have a "shift invariance" property.

(4.3) **PROPOSITION.** Suppose either that $g \in (\underline{E}^{\ell+1})^+$ or $EY_1(|g|) < \infty$. Then,

$$(4.4) \qquad E\{\sum_{k=T_1}^{T_2-1} g(X_k, \ldots, X_{k+\ell})\} = E\{\sum_{k=T_1+j}^{T_2+j-1} g(X_k, \ldots, X_{k+\ell})\}$$

for any $j \geq 0$.

Proof. Since U_k is weakly regenerative (see proof of (4.1)), we have that

$$(4.5) \qquad \sum_{k=T_1+j}^{T_{n+1}-1+j} g(X_k, \ldots, X_{k+\ell})/n \to EY_1(g) \qquad \text{a.s.}$$

if $g \in b\underline{E}^{\ell+1}$. Now, the left-hand side of (4.5) is dominated by $\|g\|(T_{n+1}-T_1)/n$, which converges to $\|g\|E\tau_1$. But

$$E(T_{n+1}-T_1)/n = E\tau_1$$

and hence $\{(T_{n+1}-T_1)/n;\ n \geq 1\}$ is uniformly integrable. It follows that the left-hand side of (4.5) is uniformly integrable, and thus we may take expectations of both sides in (4.5) (see CHUNG (1974), p. 97). Application of (2.5)(ii) then proves (4.4) for $g \in b(\underline{E}^{\ell+1})$. *A standard approximation argument gives* (4.4) in the general case. ∎

We now define the associated stationary process Y_n. For $A \in \underline{E}^{\infty}$, define the measure P^* on E^{∞} by the formula

$$P^*(A) = E\{\sum_{k=T_1}^{T_2-1} I_A(X_k, X_{k+1}, \ldots)\}/E\tau_1 \quad .$$

Let Y_n be the coordinate projection of E^∞ onto its n'th coordinate, for $n \geq 0$.

(4.6) THEOREM. The process $\{Y_n: n \geq 0\}$ is stationary with respect to the probability P^*.

Proof. It is sufficient to prove that

$$P\{(Y_0, \ldots, Y_\ell) \in A\} = P\{(Y_j, \ldots, Y_{j+\ell}) \in A\} \quad \text{for all } A \in \underline{E}^{\ell+1} \ .$$

But this result follows immediately from Proposition 4.3. ∎

In this case where $\{X_n: n \geq 0\}$ is regenerative, it is particularly simple to prove that the stationary process $\{Y_n\}$ can be constructed directly from the $\{X_n\}$ process, in the following sense. We assume that $(\Omega,\underline{G},P)$ is a sufficiently rich probability space that it supports r.v.'s β_1, β_2 independent of $\{(X_n,T_k): n,k \geq 0\}$ with distribution

$$P\{\beta_1 = k, \beta_2 = j\} = P\{\tau_1 = k\}/E\tau_1 \, I_{\{j \leq k\}}(j) \quad .$$

Let $\eta(k) = \inf\{n \geq 1: \tau_n = k\}$ and put $\alpha = T_{\eta(\beta_1)} + \beta_2$.

(4.7) **PROPOSITION.** **If $\{X_n\colon n \geq 0\}$ is regenerative, then the process $\{X_{\alpha+k}\colon k \geq 0\}$ has the same distribution on $\underline{E}^\infty$ as $\{Y_k\colon k \geq 0\}$.**

Proof. For $B \in \underline{E}^{\ell+1}$,

(4.8) $$P\{(X_\alpha, \ldots, X_{\alpha+\ell}) \in B\}$$

$$= \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} P\{X_{\eta(k)+j}, \ldots, X_{\eta(k)+j+\ell}) \in B\}\, P\{\tau_1 = k\}/E\tau_1 \; .$$

Now,

(4.9) $$P\{(X_{\eta(k)+j}, \ldots, X_{\eta(k)+j+\ell}) \in B\}$$

$$= \sum_{n=1}^{\infty} P\{(X_{T_n+j}, \ldots, X_{T_n+j+\ell}) \in B;\; \tau_1 \neq k, \ldots, \tau_{n-1} \neq k, \tau_n = k\}$$

$$= \sum_{n=1}^{\infty} P\{(X_{T_1+j}, \ldots, X_{T_1+j+\ell}) \in B;\; \tau_1 = k\}\, P\{\tau_1 \neq k\}^{n-1}$$

$$= P\{(X_{T_1+j}, \ldots, X_{T_1+j+\ell}) \in B;\; \tau_1 = k\}/P\{\tau_1 = k\} \quad ,$$

where the second equality follows from (2.1)(i) and (2.5)(ii). Substituting (4.9) into (4.8) and using Fubini proves that

$$P\{(X_\alpha, \ldots, X_{\alpha+\ell}) \in B\}$$

$$= \sum_{n=1}^{\infty} \sum_{j=0}^{k-1} P\{(X_{T_1+j}, \ldots, X_{T_1+j+\ell}) \in B;\ \tau_1 = k\}/E\tau_1$$

$$= \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} P\{(X_{T_1+j}, \ldots, X_{T_1+j+\ell}) \in B;\ \tau_1 = k\}/E\tau_1$$

$$= \sum_{j=0}^{\infty} P\{(X_{T_1+j}, \ldots, X_{T_1+j+\ell}) \in B;\ \tau_1 > j\}/E\tau_1$$

$$= E\{\sum_{j=T_1}^{T_2-1} (X_{T_1+j}, \ldots, X_{T_1+j+\ell}) \in B\}/E\tau_1 \quad ,$$

proving the result. ∎

As one might expect, the process $\{X_{\alpha+k}: k \geq 0\}$ is regenerative *with respect to the random times* $T_0' = \alpha$, $T_k' = T_{\eta(\beta_1)+k}$, $k \geq 1$. Furthermore, as is easily checked,

$$(4.10) \quad P\{(X_{T_1'}, \ldots, X_{T_1'+k}) \in B;\ \tau_1' = j_0, \ldots, \tau_{\ell+1}' = j_\ell\}$$

$$= P\{(X_{T_1}, \ldots, X_{T_1+k}) \in B;\ \tau_1 = j_0, \ldots, \tau_{\ell+1} = j_\ell\}$$

where $\tau_n' = T_{n+1}' - T_n'$. However, the first epoch of $\{X_{\alpha+n}: n \geq 0\}$ has a markedly different distribution from that of $\{X_n: n \geq 0\}$. Observe that for $B \in \underline{E}^{\ell+1}$, we have

(4.11) $P\{(X_{T_0'}, \ldots, X_{T_0'+\ell}) \in B;\ \tau_0' = \ell+1\}$

$$= \sum_{k=\ell+1}^{\infty} P\{(X_{\eta(k)+k-\ell-1}, \ldots, X_{\eta(k)+k-1}) \in B\}\ P\{\tau_1 = k\}/E\tau_1$$

$$= \sum_{k=\ell+1}^{\infty} P\{(X_{T_1+k-\ell-1}, \ldots, X_{T_1+k-1}) \in B;\ \tau_1 = k\}/E\tau_1$$

$$= P\{(X_{T_2-\ell-1}, \ldots, X_{T_2-1}) \in B;\ \tau_1 \geq \ell+1\}/E\tau_1 \quad .$$

This result will prove useful in Section 6, when we shall examine mixing conditions for regenerative processes. Formula (4.11) also leads directly to the observation that a stationary regenerative process $\{X_n: -\infty < n < \infty\}$ has the same distribution as its time reversal $\{Z_n: -\infty < n < \infty\}$ if and only if for all $B \in \underline{E}^{\ell+1}$,

(4.12) $$P\{(X_{T_1+\ell}, \ldots, X_{T_1}) \in B;\ \tau_1 \geq \ell+1\}$$

$$= P\{(X_{T_2-\ell-1}, \ldots, X_{T_2-1}) \in B;\ \tau_1 \geq \ell+1\} \quad .$$

5. Total Variation Convergence for Regenerative Processes

By Theorem 3.1 and the bounded convergence theorem, it follows that

(5.1) $$\sum_{k=0}^{n} Ef(X_k)/n \to \pi f$$

for all $f \in b\underline{E}$. One of our main goals in this section is to show that the convergence in (5.1) is uniform over $f \in b\underline{E}$ such that $\|f\| \leq 1$, and that, under certain conditions, (5.1) holds without averaging. Such uniform convergence is equivalent to total variation convergence of the corresponding measures.

In analogy with Markov chains, we say that a weakly regenerative process $\{X_n: n \geq 0\}$ is _periodic_ with period p if the span of the distribution of τ_1 is p. If the period is 1, then $\{X_n\}$ is said to be _aperiodic_. For $g \in b(\underline{E}^\infty)$, put

$$v(g;n) = Eg(V_{T_1+n}) \qquad \text{for } n \geq 0 .$$

(5.2) **THEOREM.** Let $\{X_n\}$ be an aperiodic positive recurrent regenerative process. Then, there exist constants $\gamma_n \to 0$ such that

$$\sup_{\|g\| \leq 1} \left| v(g;n) - E^*g(Y) \right| = \gamma_n$$

where E^* denotes expectation with respect to P^*. If $E\tau_1^\kappa < \infty$, then $\gamma_n = o(n^{1-\kappa})$.

Proof. We start with a renewal argument, namely

$$(5.3)\quad v(g;n) = \sum_{k=1}^{n} E\{g(V_{T_1+n});\ T_2 = T_1+k\} + E\{g(V_{T_1+n});\ T_2 > T_1+n\}$$

$$= \sum_{k=1}^{n} E\{g(V_{T_1+n-k})\}\ P\{\tau_1 = k\} + E\{g(V_{T_1+n});\ \tau_1 > n\}$$

$$\equiv \sum_{k=1}^{n} v(g;\ n-k)\ P\{\tau_1 = n-k\} + a(g;n) \quad ,$$

when the second equality is by Proposition 2.9 (i). The solution of this renewal equation is

$$(5.4)\qquad v(g;n) = \sum_{j=0}^{n} a(g;\ n-j)\ u_j$$

where $u_j = \sum_k P\{\tau_1 + \cdots + \tau_k = j\}$ (see KARLIN and TAYLOR (1975), p. 184). The renewal theorem (FELLER (1950), p. 330) asserts that

$$(5.5)\qquad v(g;n) \to \sum_{j=0}^{\infty} a(g;j)/E\tau_1 \quad .$$

Note that $a(|g|;\ j) \leq \|g\|\ P\{\tau_1 > j\}$, so, by Fubini's theorem,

$$(5.6)\quad \sum_{j=0}^{\infty} a(g;j) = E\{\sum_{j=0}^{\infty} g(V_{T_1+j})\ I_{\{\tau_1>j\}}\}$$

$$= E\{\sum_{j=T_1}^{T_2-1} g(V_j)\} = E^*\ g(Y)\cdot E\tau_1 \quad .$$

Combining (5.4) through (5.6) shows that

(5.7) $\left|v(g;n) - E^*g(Y)\right|$

$$\leq \|g\| \sum_{j=0}^{n} P\{\tau_1 > j\}\left|u_{n-j} - (E\tau)^{-1}\right| + \|g\| \sum_{j=n+1}^{\infty} P\{\tau_1 > j\} .$$

We now use an estimate of GEL'FOND (1964) for the error in the renewal theorem, namely

$$u_n = 1/E\tau + o(n^{1-\kappa}) \quad , \tag{5.8}$$

which is valid under the assumption $E\tau_1^\kappa < \infty$. This implies that

$$K_x = \sup\{\left|u_j - 1/E\tau\right| : j \geq x\} = o(x^{1-\kappa}) \quad .$$

Substituting this relation in (5.7) yields

$$\left|v(g;n) - E^*g(Y)\right|$$

$$\leq \|g\|\left(\sum_{j\geq n/2}^{\infty} P\{\tau_1 > j\} + K_{n/2}\right)$$

$$\leq \|g\|\left(n^{1-\kappa} \sum_{j\geq n/2} j^{\kappa-1} P\{\tau_1 > j\} + K_{n/2}\right)$$

$$= \|g\|\left(n^{1-\kappa}\ o(1) + o(n^{1-\kappa})\right) = \|g\|\ o(n^{1-\kappa}) \quad ,$$

the second-last equality because $E\tau_1^\kappa < \infty$. ∎

Theorem 5.2 considers the time-dependent behavior of X_n for $n \geq T_1$. Our next result deals with X_n over semi-infinite time intervals with deterministic time origin.

(5.9) **PROPOSITION.** If $\{X_n: n \geq 0\}$ is an aperiodic positive recurrent regenerative process, then

$$\left|E\{g(V_{k+n})\left|\underline{F}_0^k\right.\} - E^*g(Y)\right|$$

$$\leq \|g\|(1 + \gamma(0))\ P\{T_{\ell(k)+1} - k > n/2\left|\underline{F}_0^k\right.\} + \|g\|\ \gamma(n/2) \quad \text{a.s.,}$$

where $\gamma(x) = \sup\{\gamma(j): j \geq x\}$ for $x \geq 0$ (put $\gamma(x) = 0$ for $x < 0$).

Proof. Let $\hat{g} = g - E^*g(Y)$, and note that

$$\left|E\{\hat{g}(V_{k+n})\left|\underline{F}_0^k\right.\}\right|$$

$$\leq \left|E\{g(V_{k+n}): T_{\ell(k)+1} \leq k+n\left|\underline{F}_0^k\right.\}\right| + \|g\|\ P\{T_{\ell(k)+1} - k > n\left|\underline{F}_0^k\right.\} .$$

For the first term, we use Proposition 2.9(ii) and Theorem 5.2 to obtain

$$\left|E\{v(\hat{g};\ n + k - T_{\ell(k)+1});\ T_{\ell(k)+1} \le k+n \big| \underline{F}_0^k\}\right|$$

$$\le \|g\|\ E\{\gamma(n + k - T_{\ell(k)+1});\ T_{\ell(k)+1} - k \le n \big| \underline{F}_0^k\}$$

$$\le \|g\|\ E\{\gamma(n + k - T_{\ell(k)+1});\ T_{\ell(k)+1} - k \le n/2 \big| \underline{F}_0^k\}$$

$$+ \|g\|\ \gamma(0)\ P\{T_{\ell(k)+1} - k > n/2 \big| \underline{F}_0^k\}$$

$$\le \|g\|\ (\gamma(n/2) + \gamma(0)\ P\{T_{\ell(k)+1} - k > n/2 \big| \underline{F}_0^k\})$$

proving the proposition. ∎

It should be noted that the asymptotic nature of γ_n under $E\tau_1^\kappa < \infty$ is inherited by $\gamma(x)$, namely $\gamma(x) = 0(x^{1-\kappa})$.

(5.10) COROLLARY. (i) If $\{X_n\}$ is an aperiodic positive recurrent regenerative process, there exist constants $a_n \downarrow 0$ such that

$$\text{(5.11)} \qquad \sup_{\|g\| \le 1} \left|Eg(V_n) - E^*g(Y)\right| = a_n \ .$$

The constants a_n are dominated by $(1+\gamma(0))P\{T_1 > n/2\} + \gamma(n/2)$.

(ii) If $(E,\underline{E})$ is a separable metric space, the process $(X_n, X_{n+1}, \ldots)$ converges weakly to the stationary process Y in the product topology on E^∞.

Proof. Part (i) follows immediately from Proposition 5.9. For part (ii), it is necessary only to realize that for a separable metric space E, the product field $\underline{E}^\infty$ coincides with the Borel sets under the product topology (see DELLACHERIE and MEYER (1978), p. 9). ∎

The periodic case can be reduced to the aperiodic situation above, without difficulty. If $\{X_n: n \geq 0\}$ is a periodic positive recurrent regenerative process with period p, set $U_0 = (X_0, \ldots, X_{T_1-1})$ and put

$$U_i = (X_{T_1+(i-1)p}, \ldots, X_{T_1+ip-1}) .$$

The process $\{U_i: i \geq 0\}$ is regenerative with respect to the random times $T_0' = 0$, $T_1' = 1$, $T_2' = T_1' + \tau_1/p, \ldots, T_n' = T_{n-1}' + \tau_{n-1}/p$. Clearly, the distribution of $\tau_i' = \tau_i/p$ has span 1, and hence $\{U_i\}$ is aperiodic. We therefore immediately obtain the following generalization of Proposition 5.9.

(5.12) PROPOSITION. If $\{X_n: n \geq 0\}$ is a positive recurrent regenerative process, then there exist constants $b_n \downarrow 0$ such that

$$\sup_{\|g\| \leq 1} \left| \sum_{j=0}^{n} E\{g(V_{k+j}) | \underline{F}_0^k\}/n - E^*g(Y) \right| = b_n .$$

These results have immediate applications to general state space Markov chains (see p. 8-25 of REVUZ (1975) for definition, notation, and basic properties). A Markov chain $\{W_n: n \geq 0\}$, with transition kernel Q defined on measurable space $(E,\underline{E})$, is called Harris recurrent if there exists a set $A \in \underline{E}$, an integer k, and a probability ϕ on $(E,\underline{E})$ for which

$$(5.13)\quad \begin{array}{lll} \text{(i)} & Q_x\{W_n \in A \ \text{ i.o.}\} = 1 & \text{for all } x \\ \text{(ii)} & Q^k(x,\cdot) \geq \lambda\phi(\cdot) & \text{for all } x \in A, \text{ where } \lambda \text{ is positive.} \end{array}$$

In the case where $k = 1$, ATHREYA and NEY (1978) and NUMMELIN (1978) have shown that for each μ on $(E,\underline{E})$ the process $\{W_n\}$ can be embedded, with marginal distribution Q_μ, in a probability space $(\Omega,\underline{G},\hat{Q}_\mu)$ for which $\{W_n\}$ is regenerative. Furthermore, the regenerative process $\{W_n\}$ is positive recurrent if and only if Q has a unique invariant probability π, in which case π coincides with that given by (3.18). The basic idea behind the regenerative embedding is to "split" Q as

$$(5.14)\qquad Q(x,\cdot) = \lambda\phi(\cdot) + (1-\lambda)\,R(x,\cdot)$$

over $x \in A$. A transition out of A is distributed with probability λ as $\phi(\cdot)$ (a regeneration) and with probability $1-\lambda$ as $R(x,\cdot)$.

Proposition 5.9 provides an easy proof of the following result (see ATHREYA and NEY (1978)).

(5.15) PROPOSITION. **If $\{W_n\}$ is a Harris chain with invariant probability π, and if $k = 1$ in (5.13), then**

$$\sup_{\|g\| \leq 1} \left| E_\mu\, g(W_n) - \pi g \right| \to 0 \qquad \text{for all } \mu .$$

The general convergence results for Harris chains can be obtained from (5.15) in a reasonably straightforward way; see ATHREYA and NEY (1977).

6. Mixing Conditions for Regenerative Processes

We say that a process $\{X_n\}$ is <u>strong mixing</u> if

(6.1)
$$\sup_k \sup_{\substack{A \in \underline{F}_0^k \\ B \in \underline{F}_{k+n}^\infty}} \left| P(AB) - P(A)\,P(B) \right| = \alpha(n) \to 0 .$$

The process $\{X_n\}$ is <u>uniformly strong mixing</u> (or ϕ-mixing) if

(6.2)
$$\sup_k \sup_{\substack{A \in \underline{F}_0^k \\ B \in \underline{F}_{k+n}^\infty}} \left| P(AB) - P(A)\,P(B) \right| \leq \phi(n)\,P(A) ,$$

where $\phi(n) \to 0$.

(6.3) THEOREM. (i) If $\{X_n\}$ is an aperiodic positive recurrent regenerative process, then $\{X_n\}$ is strong mixing.

(ii) If, in addition to the above hypotheses, there exists $c(x) \downarrow 0$ such that

$$\sup_k P\{T_{\ell(k)+1} - k > n \mid \underline{F}_0^k\} \leq c(n) \qquad \text{a.s.},$$

then the process $\{X_n\}$ is uniformly strong mixing.

<u>Proof</u>. For (i), let W be a nonnegative bounded $\underline{F}_0^k$ measurable r.v., and take $g \in b\underline{E}^\infty$. Then, by Proposition 5.9,

$$(6.4)\quad EW\hat{g}(V_{k+n}) = E\{WE\{\hat{g}(V_{k+n}) \mid \underline{F}_0^k\}\}$$

$$\leq \|W\|\,\|g\|\,((1+\gamma(0))\,P\{T_{\ell(k)+1}-k > n/2\} + \gamma(n/2)) .$$

Of course,

$$(6.5)\qquad \left|EWg(V_{k+n}) - EWEg(V_{k+n})\right|$$

$$\leq \left|EW\hat{g}(V_{k+n})\right| + EW\left|Eg(V_{k+n}) - E^*g(Y)\right|$$

$$\leq \left|EW\hat{g}(V_{k+n})\right| + \|W\|\,\|g\|a_n .$$

the second inequality by (5.11). Using (6.4) and (6.5), it is clear that the proof of (6.3)(i) is therefore complete, provided that we can show that

$$P\{T_{\ell(k)+1} - k > n\} \to 0 \qquad \text{as} \quad n \to \infty ,$$

uniformly in k. Now, observe that

$$(6.6) \qquad P\{T_{\ell(k)+1} - k > n\}$$
$$\leq P\{\tau_0 > k+n\} + P\{T_{\ell(k)+1} - k > n;\ \ell(k) > 0\} \ .$$

The second term can be written as

$$(6.7) \qquad \sum_{j=1}^{k} P\{T_{\ell(k)+1} - k > n;\ T_1 = j\}$$
$$= \sum_{j=1}^{k} P\{T_{\ell(T_1+k-j)+1} - (T_1+k-j) > n;\ T_1 = j\}$$
$$= \sum_{j=1}^{k} P\{T_{\ell(T_1+k-j)+1} - (T_1+k-j) > n\}\ P\{T_1 = j\}$$
$$\leq \max_{1 \leq j \leq k} P\{T_{\ell(T_1+j)} - (T_1+j) > n\} \ .$$

The final term in (6.7) is amenable to a renewal argument, which shows that

$$(6.8) \qquad P\{T_{\ell(T_1+j)} - (T_1+j) > n\} = \sum_{k=0}^{j} P\{\tau_1 > n+j-k\}\ u_k$$

where the right-hand side of (6.8) converges to $\sum_{k\geq 0} P\{\tau_1 > k+n\}/E\tau_1$. The boundedness of $P\{\tau_1 > n+j-k\}$ allows us to use an argument similar to that of Theorem 5.2, to prove that (6.8) converges to 0 as $n \to \infty$, uniformly in j, completing part (i).

For (ii), write

$$(6.9) \qquad \left|EWg(V_{k+n}) - EWEg(V_{k+n})\right|$$

$$\leq \left|E(WE\{\hat{g}(V_{k+n})|\underline{F}_0^k\})\right| + EW\left|Eg(V_{k+n}) - E^*g(Y)\right|$$

$$\leq \|g\|\ EW((1+\gamma(0))\ c(n/2) + \gamma(n/2) + a_n) \quad . \qquad \blacksquare$$

We now examine the form of the strong mixing constants $\alpha(n)$ for the stationary regenerative process $\{Y_n: n \geq 0\}$.

(6.10) **PROPOSITION.** **If $\{X_n\}$ is an aperiodic positive recurrent regenerative process with $E\tau_1^{\kappa} < \infty$, then $\{Y_n\}$ is strong mixing with constants $\alpha(n) = o(n^{1-\kappa})$.**

Proof. Let $V_k = (Y_k, Y_{k+1}, \ldots)$ and observe that $Eg(V_k) = Eg(V_0) = E^*g(Y)$, by stationarity. Thus,

$$\left|EWg(V_{k+n}) - EWEg(V_{k+n})\right|$$

$$\leq \|W\|\ \|g\|\ (1+\gamma(0))\ P\{T'_{\ell(k)+1} - k > n/2\} + \gamma(n/2)$$

(see (6.4), (6.5), and (4.10)). We now use (6.6) to bound $P\{T'_{\ell(k)+1} - k > n/2\}$, noting that

$$P\{\tau_0' > k+n\} \leq \sum_{j=n}^{\infty} P\{\tau_1 \geq j\}/E\tau_1 \leq n^{1-\kappa} \sum_{j=n}^{\infty} j^{\kappa-1} P\{\tau_1 \geq j\}/E\tau_1$$

$$= o(n^{1-\kappa}) \quad .$$

For the second term in (6.6), we recall that $u_n = 1/E\tau + o(n^{1-\kappa})$ and apply the argument of Theorem 5.2 to (6.8); the resulting rate of convergence bounds the second term through (6.7). ∎

As in Section 5, Markov chains provide an interesting class of examples.

(6.11) PROPOSITION. Set $\{W_n: n \geq 0\}$ be a Harris chain, possessing an invariant probability π, which is aperiodic as a Markov chain. Then, $\{W_n\}$ is strong mixing for arbitrary initial distribution μ.

Proof. Let A and k be as in (5.13). Because $\{W_n: n \geq 0\}$ is an aperiodic Markov chain, it follows that W_{nk} visits A infintely often Q_μ a.s. (see Lemma 2.1 of [27]), and hence $\{W_{nk}\}$ is regenerative under a measure $\hat{Q}_\mu$ consistent with Q_μ. Assuming that the regenerative process $\{W_{nk}\}$ has period p, let $X_n = \{W_{ns}; n \geq 0\}$ where $s = pk$. The skeleton X_n is a positive recurrent regenerative process, since W_{ns} is a Harris chain with

invariant probability, and therefore, by Theorem 6.3(i), strong mixing. Let $m = as+j$, $n = bs+\ell$, where $0 \leq j, \ell < s$ and $a < b$ and consider

$$\hat{Q}_\mu(BC) - \hat{Q}_\mu(B)\,\hat{Q}_\mu(C) \tag{6.12}$$

where $B \in \underline{B}(W_0, \ldots, W_m)$ and $C \in \underline{B}(W_n, W_{n+1}, \ldots)$. Using the Markov property of W_n, it is easily seen that these two sub-σ-fields are conditionally independent given $\underline{B}(W_{(a+1)s}, W_{bs})$ (see Theorem 45, p. 36, of [12]) and hence

$$\begin{aligned}
&\hat{Q}_\mu\{BC \mid W_{(a+1)s}, W_{bs}\} \\
&\quad = Q_\mu\{B \mid W_{(a+1)s}, W_{bs}\}\, Q_\mu\{C \mid W_{(a+1)s}, W_{bs}\} \\
&\quad = Q_\mu\{B \mid W_{(a+1)s}\}\, Q_\mu\{C \mid W_{bs}\}
\end{aligned}$$

(using the Markov property of reversed chains). Thus, (6.12) can be written as the covariance between functions of X_{a+1} and X_b, and so the strong mixing result for $\{X_n\}$ can be directly applied to obtain (6.1) for $\{W_n\}$. Furthermore, it is clear that the mixing constants for $\{W_n\}$ are given by $\alpha(n/s)$, where $\alpha(n)$ are the constants for $\{X_n\}$. ∎

Proposition 6.11 gives a regenerative process proof of Theorem 1 of DAVYDOV (1973). Furthermore, our approach allows us to obtain

estimates for the strong mixing constants for stationary aperiodic Harris chains. In particular, by applying Proposition 6.9 and 6.10, we see that $\alpha(n) = o(n^{1-\kappa})$, provided $E\tau_1^\kappa < \infty$, where τ_1 is a r.v. with the distribution of the Athreya-Ney-Nummelin regeneration time.

It is also worth applying our regenerative results to Doeblin recurrent Markov chains. A chain $\{W_n\}$ is said to be Doeblin if there exists $\varepsilon > 0$ and a finite nontrivial measure ν such that

$$\nu(B) \leq \varepsilon \Longrightarrow Q^k(x,B) \leq 1-\varepsilon \quad ,$$

for some k (see DOOB (1953) for a complete discussion). If $\underline{E}$ is a separable σ-field (i.e., countably generated) and if $\{W_n\}$ has a single ergodic set (see p. 209 of [14] for definitions), then $\{W_n\}$ is Harris recurrent (see Lemma 4.6 of [18]).

(6.13) PROPOSITION. Let $\{W_n\}$ be an aperiodic Doeblin chain with single ergodic set, and assume that $\underline{E}$ is separable. Then, $\{W_n\}$ is uniformly strong mixing with uniform mixing constants $\phi(n) = o(n^{-k})$, for any $k \geq 1$.

Proof. We apply Lemma 4.7 of [18], which shows that there exists m such that

$$\sup_{x \in E} Q_x\{T_A > m\} < 1 \quad ,$$

where A is as in (5.13). Then, using the argument of Theorem 4.1 of [1] shows that

(6.14) $$\sup_{x \in E} E_x\{T_{\ell(k)+1} - k > n\} = O(\rho^n)$$

when $0 < \rho < 1$ and the T_i's are the regeneration times of W_{ns} (see proof of (6.10)). The proof of uniform strong mixing is completed by using (6.14) in conjunction with the Markov property and Theorem 6.3(ii). To obtain the estimates for $\phi(n)$, one applies (6.14) to (5.2), (5.11), and (6.9). ∎

The above result can also be found in [11] (Theorem 2), proved by a different method.

7. The Central Limit Theorem for Weakly Regenerative Processes

Let $\{X_n: n \geq 0\}$ be a positive recurrent weakly regenerative process, and let f be a real-valued $\underline{E}$-measurable function. We now wish to investigate the behavior of "normalized" sums of $f(X_n)$'s.

Assume that $\pi|f| < \infty$ (π given by (3.8)), and let $\hat{f}(X_n) = f(X_n) - \pi f$.

(7.1) THEOREM. Let $\{X_n\}$ be a positive recurrent weakly regenerative process, and suppose that $0 < \sigma^2(Y_1(|\hat{f}|)) < \infty$. Then,

(7.2) $$\sum_{k=0}^{n} \hat{f}(X_k)/n^{1/2} \Rightarrow N(0, \sigma^2)$$

where

$$\sigma^2 \, E\tau_1 = \sigma^2(Y_1(\hat{f})) + 2 \sum_{k=1}^{m} \mathrm{cov}(Y_1(\hat{f}), Y_{k+1}(\hat{f})) \quad .$$

Proof. On $\{T_1 \leq n\}$, we can write

(7.3) $$\sum_{k=0}^{n} \hat{f}(X_k)/n^{1/2} = Y_0(\hat{f})/n^{1/2} + \sum_{k=1}^{\ell(n)-1} Y_k(\hat{f})/n^{1/2} + R_n(\hat{f})/n^{1/2}$$

where

$$R_n(\hat{f}) = \sum_{k=T_\ell(n)}^{n} \hat{f}(X_k)/n^{1/2} \quad .$$

The term $R_n(\hat{f})$ can be bounded as follows:

(7.4) $$\left|R_n(\hat{f})\right| \leq \max\{Y_k(|\hat{f}|)\colon 0 \leq k < n(m+1)\}/n^{1/2}$$

$$= \max_{0 \leq j < m+1} \max\{Y_{(m+1)k+j}(|\hat{f}|)\colon 0 \leq k \leq n-1\}/n^{1/2} \quad .$$

Since $\{Y_n(|\hat{f}|)\}$ is m-dependent, it follows tht $\{Y_{(m+1)k+j}(|\hat{f}|)\colon k \geq 0\}$ is an i.i.d. sequence of r.v.'s with common finite variance.

Thus, by the Borel-Cantelli lemma,

$$Y_{(m+1)k+j}(|\hat{f}|)/k^{1/2} \to 0 \qquad \text{a.s.}$$

Hence, for each $\varepsilon > 0$, there exists $n(\varepsilon)$ such that for all $k \geq n(\varepsilon)$,

$$Y_{k(m+1)+j}(|\hat{f}|)/k^{1/2} < \varepsilon \ .$$

So, for $k \geq n(\varepsilon)$

$$\overline{\lim} \max_{0 \leq i \leq k} Y_{i(m+1)+j}(|\hat{f}|)/k^{1/2}$$

$$\leq \overline{\lim} \max_{0 \leq i \leq n(\varepsilon)} Y_{i(m+1)+j}(|\hat{f}|)/k^{1/2}$$

$$+ \overline{\lim} \max_{n(\varepsilon) \leq i \leq k} Y_{i(m+1)+j}(|\hat{f}|)/i^{1/2}$$

$$< \varepsilon \ ,$$

and thus $|R_n(\hat{f})| \to 0$ a.s. We now use a technique employed by CHUNG (1967) to deal with the second term in (7.3). Let

$$b(k) = \max\{j: j(m+1)\ E\tau_1 \leq k\}$$

and put $k' = b((1-\varepsilon^2)n)$, $k^* = b(n)$, $k'' = b((1+\varepsilon^2)n)$ for $\varepsilon > 0$. Since $\ell(n)/n \to 1/E\tau_1$ a.s. (this can be proved by averaging as in (3.3)), it must be that there exists $n(\varepsilon)$ such that

$$\Lambda = \{k'(m+1) < \ell(n)-1 \leq k''(m+1) \quad \text{for all} \quad n \geq n(\varepsilon)\}$$

has probability at least $1-\varepsilon$. On Λ,

$$\left|\sum_{j=1}^{\ell(n)-1} Y_j(\hat{f}) - \sum_{j=1}^{k^*(m+1)} Y_j(\hat{f})\right|$$

$$\leq 2 \sum_{i=0}^{m} \max_{k' < j \leq k''} \left|\sum_{p=k'}^{j} Y_{p(m+1)+i}(\hat{f})\right| .$$

The Kolmogorov inequality applies to each individual "max" term, and thus it follows that

$$\max_{k' < j \leq k''} \left|\sum_{p=k'}^{j} Y_{p(m+1)+i}(\hat{f})\right| / k^{*1/2} \to 0$$

in probability. Hence,

$$\left(\sum_{j=1}^{\ell(n)-1} Y_j(\hat{f}) - \sum_{j=1}^{k^*(m+1)} Y_j(\hat{f})\right) / k^{*1/2} \to 0$$

in probability. Applying the central limit theorem for m-dependent sequences (see [3], p. 174) shows that

$$\sum_{j=1}^{k^*(m+1)} Y_j(\hat{f})/k^{*1/2} \Longrightarrow N(0, \sigma^2 E\tau_1(m+1)) \quad .$$

The proof is finished by using the "converging together" lemma (see p. 25 of [3]) and observing that

$$k^*/n \to 1/(m+1) E\tau_1 \quad . \quad \blacksquare$$

It is worth pointing out that in the regenerative case, Chung's proof (see p. 100 of [6]) shows that the central limit theorem (CLT) holds under the slightly weaker assumption that $0 < \sigma^2(Y_1(\hat{f})) < \infty$. Theorem 7.1 also leads to a new CLT for Harris chains, in light of the fact that Harris chains are weakly regenerative (see [19], Proposition 4.11). For other versions of the Harris chain CLT, see OREY (1959), COGBURN (1970), and MAIGRET (1978).

As is well known in stationary process theory, the variance constant σ^2 for partial sums of the form (7.2), coming from a stationary process $\{Y_i\}$, is generally given by

(7.5) $$\sigma^2(\hat{f}(Y_0)) + 2 \sum_{k=1}^{\infty} \operatorname{cov}(\hat{f}(Y_0), \hat{f}(Y_k)) \quad .$$

Let $\{Y_i\}$ be a stationary positive recurrent regenerative process, and observe that, under certain moment conditions, Theorem 7.1 applies. This leads one to suspect that (7.5) is equal to $\sigma^2(Y_1(\hat{f}))/E\tau_1$. Indeed, this kind of result is frequently implicitly used in constructing consistent variance estimates for stationary processes as they arise, for example, in simulation; see FISHMAN (1978), p. 262.

(7.6) **PROPOSITION.** Suppose that $\{X_n\}$ is a positive recurrent aperiodic regenerative process with $0 < \sigma^2(Y_1(|f|)) < \infty$, $E_\pi|f(X)|^{2+\delta} < \infty$. Then, if the series (7.5) converges absolutely to a positive constant,

$$\sigma^2(\hat{f}(Y_0)) + 2\sum_{k=1}^{\infty} \operatorname{cov}(\hat{f}(Y_0), \hat{f}(Y_k)) = \sigma^2(Y_1(\hat{f}))/E\tau_1 \quad . \tag{7.7}$$

Proof. By Proposition 6.10, the process $\{Y_n\}$ is strong mixing with mixing constants $\alpha(n) = o(n^{-2})$. Then, since $(1+\delta)/(2+\delta) > 1/2$, it follows that

$$\sum_{n=1}^{\infty} \alpha(n)^{(1+\delta)/(2+\delta)} < \infty \quad .$$

Furthermore, since the left side of (7.7), call it σ_1^2, converges absolutely,

$$\sigma^2\Big(\sum_{k=0}^{n} \hat{f}(Y_k)\Big)^2/n \to \sigma_1^2 > 0$$

(see [3], p. 172), and thus Corollary 5.3 of HALL and HEYDE (1980) applies, yielding the CLT

$$\sum_{k=0}^{n} \hat{f}(Y_k)/n^{1/2} \Rightarrow N(0, \sigma_1^2) \quad .$$

On the other hand, as discussed above,

$$\sum_{k=0}^{n} \hat{f}(Y_k)/n^{1/2} \Rightarrow N(0, \sigma_2^2)$$

where $\sigma_2^2 = \sigma^2(Y_1(f))/E\tau_1$, proving the result. ∎

A corresponding theorem for the periodic case can be obtained by considering the process $\{U_i\}$ (see remarks following Corollary 5.10). Note that all countable state positive recurrent Markov chains are positive recurrent regenerative processes. Proposition 7.6, in such a context, yields a result complementary to Theorem 3, p. 102, of CHUNG (1967).

8. A Splitting Property of a Certain Class of Regeneration Times

A random time T is said to be a splitting time for the process $\{X_n: n \geq 0\}$ if for each $n \geq 1$,

$$\{T = n\} = C_n \cap D_n \qquad \text{a.s.} \tag{8.1}$$

where $C_n \in \underline{F}_0^n$, $D_n \in \underline{F}_n^\infty$ (see JACOBSEN (1974) for details). Our main goal in this section is to show that for a reasonably general class of Markov chains, a strong regeneration time must necessarily be a splitting time. This, in turn, will allow us to totally characterize the nature of the strongly regenerative chains in the class.

We start by assuming that $\{X_n\}$ is a Markov chain taking values in a measurable space $(E,\underline{E})$, where $\underline{E}$ is the class of Borel sets of the complete, separable metric space E. We shall further require that the transition kernel Q of the process $\{X_n\}$ is λ-continuous in the sense that $Q(x,\cdot)$ is absolutely continuous with respect to some fixed σ-finite reference measure $\lambda(\cdot)$, for each x in E. Hence, by Proposition 5.1 of [29], one may write Q as

$$Q(x,B) = \int_B q(x,y)\ \lambda(dy)$$

where q is jointly measurable in the product σ-field $\underline{E} \times \underline{E}$.

As is well-known in Markov chain theory, the process $\{X_n\}$ may be represented as the measurable coordinate projections on the product space $\Omega = E \times E \times \cdots$. Then, for each μ on $(E,\underline{E})$, the chain

X_n induces a probability Q_μ on Ω corresponding to the chain started with initial distribution μ. Putting $\underline{E}_0^{n-1}$, $\underline{E}_n^\infty$ equal to the σ-fields generated by the first n coordinates and the remaining coordinates, respectively, we let $Q_\mu^1(dx)$, $Q_\mu^2(dy)$ be the "marginal" measures defined by

$$(8.2)\qquad Q_\mu^1(A) = Q_\mu\{(X_0, \ldots, X_{n-1}) \in A\}, \qquad A \in \underline{E}_0^{n-1}$$

$$Q_\mu^2(A) = Q_\mu\{(X_n, \ldots, X_{n-1}) \in B\}, \qquad B \in \underline{E}_n^\infty$$

Our first result is the following.

(8.3) **PROPOSITION.** If $\{X_n: n \geq 0\}$ is λ-continuous, then there exists a jointly measurable function $f(x,y)$ such that

$$Q_\mu(dx,\, dy) = f(x,y)\, Q_\mu^1(dx)\, Q_\mu^2(dy) \quad .$$

Proof. Let $Q_z(dy)$ be the probability on Ω associated with $X_0 = z$. Then, the Markov property of $\{X_n\}$ allows one to write

$$Q_\mu(dx,\, dy) = \int_E \lambda(dz)\, Q_\mu^1(dx)\, q(x,z)\, Q_z(dy) \quad .$$

Thus, for a rectangle $A \times B$ ($A \in \underline{E}_0^{n-1}$, $B \in \underline{E}_n^\infty$), we have, by Fubini,

$$Q_\mu(A \times B) = \int_E \lambda(dz)\ Q_z(B) \int_A Q_\mu^1(dx)\ q(x,z)$$

$$= \int_E \lambda(dz)\ Q_z(B) \int_A Q_\mu^1(dx)\ q(x,z)\ I_{\{h(z)>0\}}$$

where

$$h(z) = \int_{E_0^{n-1}} Q_\mu^1(dx)\ q(x,z) \quad .$$

Hence, for any rectangle $A \times B$,

$$Q_\mu(A \times B) = Q_\mu^*(A \times B) \ , \tag{8.4}$$

where Q_μ^* is defined by

$$Q_\mu^*(dx,\ dy) = \int_E \lambda(dz)\ Q_z(dy)\ Q_\mu^1(dx)\ \frac{q(x,z)}{h(z)}\ h(z)\ I_{\{h(z)>0\}} \ . \tag{8.5}$$

This measure is clearly absolutely continuous with respect to

$$\int_E \lambda(dz)\ Q_z(dy) Q_\mu^1(dx)\ h(z) = Q_\mu^1(dx)\ Q_\mu^2(dy) \tag{8.6}$$

Since the rectangles generate the product σ-field on Ω, it follows, by (8.4), that $Q_\mu = Q_\mu^*$. Thus, the Radon-Nikodym theorem, applied to (8.5) and (8.6), concludes the proof. ▮

(8.7) **THEOREM.** Let T_k be a regeneration time for a λ-continuous Markov chain. Then, Q_μ a.s.,

$$I_{\{Q_\mu\{T_k=n|F_0^\infty\}>0\}}$$

$$= I_{\{Q_\mu\{T_k=n|F_0^{n-1}\}>0\}} \cdot I_{\{Q_\mu\{T_k=n|\underline{F}_n^\infty\}>0\}}$$

where

$$F_j^{j+k} = \underline{B}(X_j, \ldots, X_{j+k}) .$$

<u>Proof</u>. Let P^* be the probability on $(E^\infty, \underline{E}^\infty)$ defined by

$$(8.8) \qquad P^*(A) = Q_\mu\{V_0 \in A;\ T_k = n\}/Q_\mu\{T_k = n\}$$

where $V_j = (X_j, X_{j+1}, \ldots)$. Note that if $A \in \underline{E}_0^{n-1}$, $B \in \underline{E}_n^\infty$, then

$$(8.9) \quad Q_\mu\{T = n\} \cdot P^*(AB) = Q_\mu\{(X_0, \ldots, X_{n-1}) \in A,\ V_n \in B;\ T_k = n\}$$

$$= Q_\mu\{(X_0, \ldots, X_{n-1}) \in A;\ T_k = n\}\ Q_\mu\{V_{T_1} \in B\}$$

by the regenerative property. But

$$Q_\mu\{V_n \in B;\ T_k = n\} = Q_\mu\{V_{T_1} \in B\}\ Q_\mu\{T_k = n\}$$

and thus (8.8) becomes

$$P^*(AB) = P^*(A)\ P^*(B)\ . \tag{8.10}$$

Let P_1^*, P_2^* be the marginal measures of P^* on $\underline{E}_0^{n-1}$, $\underline{E}_n^\infty$ respectively, and observe that (8.10) implies that P^* equals the product measure $P_1^* \times P_2^*$ on an algebra generating $\underline{E}$. Thus,

$$P^*(C) = \int [\int I_C(x,y)\ P_1^*(dx)]\ P_2^*(dy)\ . \tag{8.11}$$

Now, $P_1^* < P_1$ and $P_2^* < P_2$ where P_1, P_2 are the marginal measures of P on $\underline{E}_0^\infty$ ($Q_1 \ll Q_2$ means that Q_1 is absolutely continuous with respect to Q_2). Hence, by the Radon-Nikodym theorem,

$$P_1^*(dx) = h_1(x)\ P_1(dx)$$

$$P_2^*(dx) = h_2(x)\ P_2(dx)$$

for appropriately measurable h_1 and h_2. So, from (8.11), we get

$$P^*(C) = \int [\int I_C(x,y)\ h_1(x)\ h_2(y)\ P_2(dy)]\ P_1(dx)\ . \tag{8.12}$$

Proposition (8.2) then shows that

$$(8.13) \qquad P'(dx, dy) = f(x,y)\, P_1(dx)\, P_2(dy) \quad .$$

We now use the fact that $P^* \ll P'$, so that

$$(8.14) \qquad P^*(dx,dy) = h(x,y)\, P'(dx, dy) \ .$$

Combining (8.12) through (8.14) proves that

$$h(x,y)\, f(x,y) = h_1(x)\, h_2(y)$$

$P_1 \times P_2$ a.s. The absolute continuity of P with respect to $P_1 \times P_2$ gives

$$(8.15) \qquad h(x,y) = h_1(x)\, h_2(y)/f(x,y)$$

P a.s. (we interpret the quotient as zero if the denominator vanishes). Now observe that

$$h((X_0, \ldots, X_{n-1}), V_n) = P\{T_k = n | \underline{F}_0^\infty\} \qquad \text{a.s.}$$

$$h_1(X_0, \ldots, X_{n-1}) = P\{T_k = n | \underline{F}_0^{n-1}\} \qquad \text{a.s.}$$

$$h_2(V_n) = P\{T_k = n | \underline{F}_n^\infty\} \qquad \text{a.s.} \quad .$$

Thus, (8.15) proves the theorem. ∎

Equation (8.16) can be used to characterize the class of strongly regenerative λ-continuous Markov chains.

(8.17) THEOREM. Let $\{X_n\}$ be a λ-continuous Markov chain, with transition kernel Q, taking values in a complete separable metric space. Then, the following are equivalent:

i) $\{X_n\}$ is a strongly regenerative process under initial distribution μ.

ii) there exist sets $A, B \in \underline{E}$ such that $Q_\mu\{(X_{n-1}, X_n) \in A \times B$ i.o.$\} = 1$ and

$$Q(x, B \cap C) = \phi(C)\, Q(x,B)$$

for all $x \in A$ and $C \in \underline{E}$, where ϕ is a measure on $(E, \underline{E})$.

Proof. We first prove that (i) implies (ii). Select n so that $Q_\mu\{T_2 = n\} > 0$, let $g \in b\underline{E}$, and consider

(8.18) $$E_\mu\{Zg(X_n);\ T_2 = n\}$$

where Z is an arbitrary bounded function measurable with respect to $\underline{G}_0^{n-1} = \underline{B}(X_0, \ldots, X_{n-1})$. By the Markov property and (8.16), (8.18) can be written as

$$
\begin{aligned}
(8.19) \qquad & E_\mu\{Zg(X_n)\, I_{\Gamma_1}\, I_{\Gamma_2}\} \\
&= E_\mu\{Z\, I_{\Gamma_1}\, g(X_n)\, E_\mu\{\Gamma_2 | \underline{G}_0^{n-1}\}\} \\
&= E_\mu\{Z\, I_{\Gamma_1}\, g(X_n)\, h(X_n)\} \\
&= E_\mu\{Z\, I_{\Gamma_1}\, E_\mu\{g(X_n)\, h(X_n) | X_{n-1}\}\} \\
&= E_\mu\{Z\, I_{\Gamma_1} (Qgh)\, (X_{n-1})\}
\end{aligned}
$$

where $h(X_n) = P\{\Gamma_2 | X_n\}$, and Qf is defined, for $f \in b\underline{E}$, by the formula

$$(Qf)(y) = \int f(z)\, Q(y,dz) \ .$$

On the other hand, the regenerative property dictates that (8.18) is equal to

(8.20) $$E_\mu\{Z;\ T_2 = n\}\ E_\mu g(X_{T_2})$$

$$= E_\mu\{Z\ I_{\Gamma_1}\ h(X_n)\ E_\mu g(X_{T_2})\}$$

$$= E_\mu\{Z\ I_{\Gamma_1}(Qh)(X_{n-1})\ E_\mu g(X_{T_2})\}\ .$$

It follows that on Γ_1, and hence on $\{T = n\}$,

(8.21) $$(Qgh)(X_{T_2-1}) = (Qh)(X_{T_2-1})\ E_\mu g(X_{T_2})$$

Q_μ a.s., for $g \in b\underline{E}$. Now, $\underline{E}$ is separable, so it is generated by a countable algebra $C_1, C_2, \ldots,$. By (8.21), $Q_\mu(\Lambda) = Q_\mu\{T_2 = n\}$, where

$$\Lambda = \{(Qgh)(X_{T_2-1}) = (Qh)(X_{T_2-1})\ E_\mu g(X_{T_2}) \text{ for } g = I_{C_k} \text{ for all } k\}\ ,$$

and thus for ν a.e. x

(8.22) $$(Qgh)(x) = (Qh)(x)\ E_\mu g(X_{T_2})$$

simultaneouly over all $g \in b\underline{E}$, where $\nu(dx) = Q_\mu\{X_{T_2-1} \in dx;\ T_2 = n\}$. Let $B = \{y\colon h(y) > 0\}$, and observe that if $g(y) = I_B(y)\ g(y)$, then (8.22) gives

(8.23) $$(Qg)(x) = (Qh)(x)\ E_\mu\ g(X_{T_2})/h(X_{T_2})\ .$$

Letting $\eta(dy) = P_\mu\{X_{T_2} \in dy\}$, we see that (8.23) yields

(8.24) $$Q(x,dy) = c_x\,\eta(dy)/h(y) \qquad \text{for } y \in B ,$$

where c_x is a constant. Let $\phi(dy) = c\cdot\eta(dy)/h(y)$, where c is a normalization which makes ϕ a probability. By (8.24), we have

(8.25) $$Q(x, C \cap B) = \hat{c}_x\,\phi(C)$$

where $\hat{c}_x = c_x/c$. Putting C equal to B reveals that $\hat{c}_x = Q(x,B)$. Finally, recall that, by absolute continuity,

$$\nu(dx) = k(x)\; Q_\mu\{X_{T_2-1} \in dx\} \quad .$$

Putting $A = \{x: k(x) > 0\}$, we see that

$$Q_\mu\{(X_{T_2-1}, X_{T_2}) \in A \times B\} > 0$$

and thus

$$Q_\mu\{(X_{n-1}, X_n) \in A \times B \text{ i.o.}\} = 1 \quad ,$$

finishing the proof of (i) implying (ii).

For the converse, let $T_0 = 0$ and put

$$T_{n+1} = \inf\{k > T_n+1: (X_{k-1}, X_k) \in A \times B\}$$

For $Z_1 \in b\underline{G}_0^{k-1}$, $g \in b\underline{E}^\infty$, the Markov property proves that

(8.26) $$E_\mu\{Z_1\ g(V_k);\ T_1 = k\}$$

$$= E_\mu\{Z_1\ g(V_k);\ X_{k-1} \in A,\ X_k \in B\}$$

$$= E_\mu\{Z_1\ E_{X_{k-1}}\{g(V_1);\ X_1 \in B\};\ X_{k-1} \in A\}$$

$$= E_\mu\{Z_1\ P_{X_{k-1}}\{X_1 \in B\};\ X_{k-1} \in A\}\ E_\phi\{g(V_0)\}$$

$$= E_\mu\{Z_1;\ T_1 = k\}\ E_\phi\{g(V_0)\} \quad .$$

Put $Z_1 = 1$ in (8.26) and sum over all k in (8.26); this shows that $E_\phi\{g(V_0)\} = E_\mu\{g(V_{T_1})\}$, and yields the independence of $\underline{H}_0$ and $\vee_{k=1}^{\infty} \underline{H}_k$. An inductive argument proves that the entire collection $\underline{H}_k$ is independent. For the identically distributed property (2.1)(ii), repetition of (8.22) for T_n proves that for all $n \geq 1$,

$$E_\phi\{g(V_0)\} = E_\mu\{g(V_{T_n})\} \quad .$$

In particular, setting

$$g(v) = I_C(v_0,\ \ldots,\ v_{k-1})\ I_A(v_{k-1})\ I_B(v_k)$$

shows that

$$E_\mu\{(X_{T_n},\ \ldots,\ X_{T_n+k-1}) \in C;\ \tau_n = k\}$$

have a common value, proving (2.1)(ii). ∎

This theorem shows that if a λ-continuous Markov chain is strongly regenerative, then one can choose the regeneration times to be stopping times with respect to the process fields. This result extends, in a certain sense, to Markov chains that are weakly regenerative; see [19]. For some related results on splitting times for countable state Markov chains, we refer the reader to JACOBSEN and PITMAN (1977).

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SOME NEW RESULTS IN REGENERATIVE PROCESS THEORY,

Peter W. Glynn

A process $\{X_n: n \geq 0\}$ is said to be weakly regenerative of order m if there exist random times $\{T_k: k \geq 1\}$ such that the random "tour" $(T_{k+1}-T_k, X_{T_k}, \ldots, X_{T_{k+1}-1})$ are identically distributed and m-dependent; this concept generalizes the notion of regenerative process in which m equals zero, and turns out to have useful simulation application. We show that such processes obey strong laws of large numbers and central limit theorems, and that the associated empirical measure converges weakly, with probability one, to a limiting "ergodic" measure π, which captures all the steady-state information inherent in the process $\{X_n\}$. For the case in which $\{X_n\}$ is regenerative; we study total variation convergence problems associated with π. As a result, we are able to obtain general conditions under which regenerative processes are strong mixing and uniformly strong mixing-extimates for the mixing constants are also given. Finally, we consider the case in which the random times T_k are measurable with respect to the process fields generated by a regenerative Markov chain $\{X_n\}$ satisfying a general continuity-type condition. In this case, we are able to totally characterize the form of such a chain's transition kernel.

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